Lumped Parameter Analysis of a Dynamic Loudspeaker

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The low frequency behaviour of dynamic loudspeaker drivers is well understood. The pioneering work of A.N.Thiele and R.H.Small brought a new understanding of enclosure design, particularly vented loudspeaker enclosure design. I have on hand the "Loudspeakers in Vented Boxes" paper by A.N.Thiele published in the IRE proceedings and have to admit that I find it hard to follow, principally because it is built around the contruction of equivalent circuits to represent driver behaviour. Being unfamiliar with the basic approach of that modelling (it isn't covered in the paper) makes it difficult to understand where there various impedances stem from. The use of an equivalent circuit also obscures the origins of particular terms in the governing equations. For that reason I chose to construct an analysis that takes a different, and I believe, more readily understood approach.

Assumptions

All low frequency models of dynamic loudspeakers generally assume that the driver operates in what is referred to as the piston operating range. This generally means that in this frequency range the driver cone behaves as a rigid body. The reality is that real drivers are never rigid bodies and it is essentially impossible to make a perfect piston but for low frequencies the rigid body assumption holds true.

At higher frequencies vibrations travel transversally along the cone surface in what is generally referred to as cone break up. The point at which cone break up becomes a dominant factor in driver response generally occurs when the wave length of the sound in air is comparable to or less than twice the cone diameter. For an 8 inch driver this occurs at around 860Hz[1]. Similarly, for the effect of the box on the response it is assumed that the wavelength of sound is large compared with the physical dimensions of the enclosure, which is generally true for the range of frequencies for which the box influences the driver response.

In this anlysis it is also assumed that the air mass loading on the cone (the mass of air in contact with the cone) and the radiation resistance (energy transmitted to air as sound waves) is lumped in with the model mass and mechanical resistance terms.

[1]This follows from the relationship between speed of propagation of sound in air V_{air}, wavelength λ and wave frequency$f$, namely $V_{air} = f\lambda$

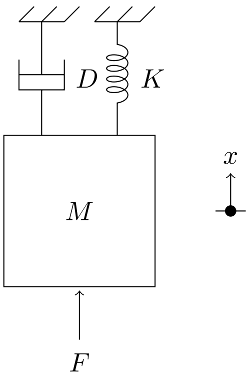


Figure 1: mechanical model of loudspeaker

The Model

We start with the rigid body assumption and consider the mechanical components responsible for the dynamic response. At its simplest level it is a mass-spring-damper system, the cone and the voice supplying the moving mass, the suspension (including the spider and ring surround) providing the stiffness and mechanical damping / resistance. This is illustrated in the lumped parameter mechanical model in figure 1.

Balancing the forces in this mechanical system we find that,

$$F = Kx + M\frac{d^2x}{dt^2} + D\frac{dx}{dt} \tag{1}$$

where F is the force applied to the voice coil, x is the displacement of the cone, K is the suspension stiffness, M is the moving mass and D is the mechanical damping or resistance. Now we define the Laplace Transform of $x(t)$ with zero intial conditions as,

$$X\left(S\right) = \mathcal{L}\left(x\left(t\right)\right) \tag{2}$$

It can be shown that,

$$\mathcal{L}\left(\frac{dx\left(t\right)}{dt}\right) = SX\left(S\right) \tag{3}$$

or more generally,

$$\mathcal{L}\left(\frac{d^nx\left(t\right)}{dt^n}\right) = S^nX\left(S\right) \tag{4}$$

This identity can then be used to transform our differential equation in the time domain into a polynomial in S, the complex frequency domain,

$$F\left(S\right) = \left(MS^2 + DS + K\right)X\left(S\right) \tag{5}$$

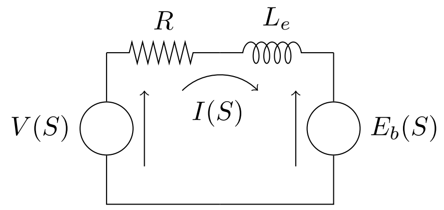


Figure 2: Excitation Circuit

This shorthand way of dealing with differential equations is common practice in electrical engineering and makes analysis considerably easier than dealing with Laplace transforms directly.

Now turning our attention to the electrical part of the the loudspeaker we can represent the excitation circuit with the structure shown in figure 2. From Ohms law[2] and noting that the complex impedance of an inductor[3]is SL it follows that,

$$I(S) = \frac{V(S) - E_b(S)}{R + SL_e} \tag{6}$$

where V is the voltage applied to the voice coil, R is the voice coil resistance, L_e is the voice coil leakage inductance and E_b is the back EMF induced in the voice coil by the motion of the cone. From the Lorentz force law we know that F is proportional to current giving

$$F(S) = K_f I(S) = K_f \frac{V(S)}{R + SL_e} - K_f \frac{E_b(S)}{R + SL_e} \tag{7}$$

where K_f is the constant of proportionality between force and current. From Faradays law we know that the back EMF E_b is proportional to cone velocity giving

$$E_b(S) = K_g S X(S) \tag{8}$$

where K_g is the constant of proptionality between back EMF and cone velocity. Substituting equation 8 into 6 and re-arranging we find,

$$\frac{X(S)}{V(S)} = \frac{K_f}{S K_g K_f + \frac{F(S)}{X(S)} (R + SL_e)} \tag{9}$$

We also note that acoustic sound pressure is proportional to the acceleration of the cone giving

$$A_{spl}(S) = K_a S^2 \tag{10}$$

[2]The voltage, v, required to create a current flow of i through a resistor of resistance R ohms is given by $v = iR$

[3]the complex impedance of an inductor follows from the differential equation governing its operation, namely $v = L\frac{di}{dt}$ where v is the voltage across the inductor and i is the current through it. Hence $Z(S) = SL$

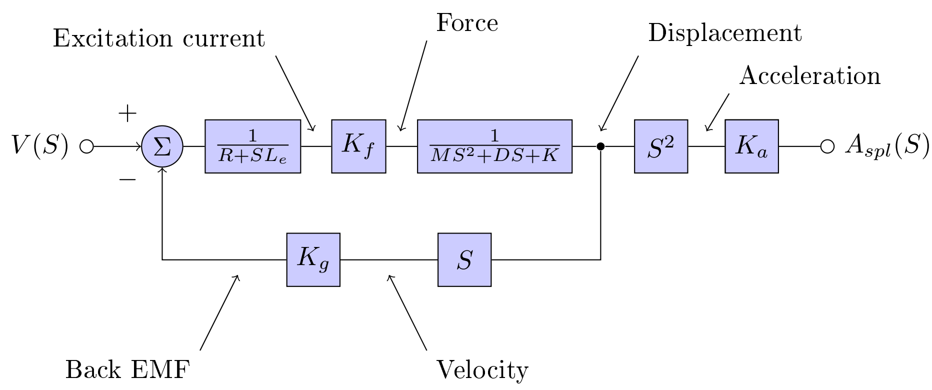


Figure 3: Dynamic Loudspeaker Block Diagram

where K_a is the constant of proportionality between acoustic sound pressure and cone acceleration and A_{spl} is the sound pressure level. With these relationships we can see that this electro-dynamic system is completely represented by the block diagram of figure 3.

Looking at the voice coil component in more detail, let us assume the magnetic induction / flux density in the pole air gap is B, that n turns of wire are within the air gap carrying a current of i and the coil has a nominal diameter of D. Then using the Lorentz force law we find,

$$F = ilB = in\pi DB \tag{11}$$

from which we can deduce that,

$$K_f = n\pi DB \tag{12}$$

From Faradays law we have,

$$E_b = n\frac{d\Phi}{dt} \tag{13}$$

where Φ is the flux cut by a single turn of wire in the voice coil. Therefore the change of flux with a change of displacement is,

$$d\Phi = \pi DBdx \tag{14}$$

giving,

$$E_b = n\pi DB\frac{dx}{dt} \tag{15}$$

From which it is clear that,

$$K_g = n\pi DB = K_f \tag{16}$$

Loudspeaker Displacement Response

From figure 3 we see that the model is a negative feedback loop. Substituting equation 5 into equation 9 and re-arranging gives the displacement response,

$$\frac{X(S)}{V(S)} = \frac{K_f}{(R + SL_e)(MS^2 + DS + K) + {K_f}^2 S} \tag{17}$$

To simplify treatment of the displacement reponse of the driver we shall ignore the effect of the leakage inductance L_e . This is of little consequence for the low frequency analysis of driver behaviour.

$$\frac{X(S)}{V(S)} = \frac{K_f}{MRS^2 + \left(DR + {K_f}^2\right) S + KR} \tag{18}$$

We can re-write this response in the normalised,

$$\frac{X(S)}{V(S)} = \left(\frac{1}{K_f \omega_s Q_e}\right) \frac{{\omega_s}^2}{S^2 + \left(\frac{\omega_s}{Q_m} + \frac{\omega_s}{Q_e}\right) S + {\omega_s}^2} \tag{19}$$

where,
$\frac{\omega_s}{Q_m} = \frac{D}{M}$
$\frac{\omega_s}{Q_e} = \frac{K_f^2}{RM}$
${\omega_s}^2 = \frac{K}{M}$
We shall ellaborate on these terms later when discussing driver impedance. The displacement response is a second order low pass response.
The displacement frequency response is obtained by substituting S=$j\omega$ giving[4],

$$\frac{X(\omega)}{V(\omega)} = \left(\frac{1}{K_f \omega_s Q_e}\right) \frac{{\omega_s}^2}{j\omega\left(\frac{\omega_s}{Q_m} + \frac{\omega_s}{Q_e}\right) + {\omega_s}^2 - \omega^2} \tag{20}$$

Loudspeaker Sound Pressure Response

The radiated sound pressure is proportional to acceleration. Referring again to figure 3 and equation 19 we see that,

$$\frac{A_{spl}(S)}{V(S)} = \left(\frac{K_a \omega_s}{K_f Q_e}\right) \frac{S^2}{S^2 + \left(\frac{\omega_s}{Q_m} + \frac{\omega_s}{Q_e}\right) S + {\omega_s}^2} \tag{21}$$

We see the sound pressure response is a second order high pass response. The sound pressure frequency response is obtained by substituting S=$j\omega$ giving,

$$\frac{A_{spl}(\omega)}{V(\omega)} = \left(\frac{K_a \omega_s}{K_f Q_e}\right) \frac{-\omega^2}{j\omega\left(\frac{\omega_s}{Q_m} + \frac{\omega_s}{Q_e}\right) + {\omega_s}^2 - \omega^2} \tag{22}$$

[4] j is the imaginary number defined by the identity $j = \sqrt{-1}$. Physicists, Mathematicians and mechanical Engineers use i to indicate the imaginary number but Electrical Engineers use j to avoid confusing it with an electric current.

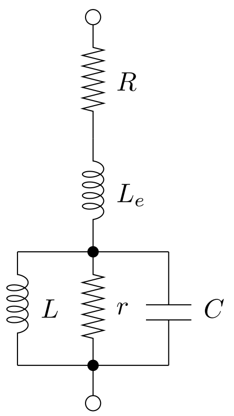


Figure 4: Impedance Presented by a Loudspeaker Mounted in an Infinite Baffle

Loudspeaker Impedance

From Ohms law,

$$Z(S) = \frac{V(S)}{I(S)} \tag{23}$$

$$I(S) = \frac{V(S) - E_b(S)}{R + SL_e} \tag{24}$$

Therefore,

$$Z(S) = \frac{V(S)}{V(S) - E_b(S)}(R + SL_e) = \frac{1}{1 - \frac{E_b(S)}{V(S)}}(R + SL_e) \tag{25}$$

Substituting equation 8 we find,

$$Z(S) = \frac{1}{1 - K_g S \frac{X(S)}{V(S)}}(R + SL_e) \tag{26}$$

Substituting the displacement response of equation 17 into the above we find,

$$Z(S) = \frac{(R + SL_e)\left(MS^2 + DS + K\right) + {K_f}^2 S}{(R + SL_e)(MS^2 + DS + K)}(R + SL_e) \tag{27}$$

$$Z(S) = \frac{S^3 L_e + S^2\left(R + \frac{L_e D}{M}\right) + S\left(\frac{RD}{M} + \frac{L_e K}{M} + \frac{{K_f}^2}{M}\right) + \frac{RK}{M}}{S^2 + \frac{D}{M}S + \frac{K}{M}} \tag{28}$$

To see how this translates into an equivalent electrical circuit, consider the network in figure 4. The impedance of this network[5] is,

[5] the complex impedance of a capacitor follows from the differential equation governing its operation, namely $i = C\frac{dv}{dt}$ where v is the voltage across the capacitor and i is the current through it. Hence $Z(S) = \frac{1}{SC}$

$$Z(S) = R + \frac{1}{\frac{1}{SL} + \frac{1}{r} + SC} + SL_e \tag{29}$$

$$Z(S) = \frac{S^3 L_e + S^2\left(R + \frac{L_e}{Cr}\right) + S\left(\frac{1}{C} + \frac{R}{Cr} + \frac{L_e}{LC}\right) + \frac{R}{LC}}{S^2 + S\frac{1}{Cr} + \frac{1}{LC}} \tag{30}$$

Comparing equation 28 and equation 30 we can see that if we choose,

$$\frac{1}{Cr} = \frac{D}{M} \tag{31}$$

$$\frac{1}{LC} = \frac{K}{M} \tag{32}$$

the denominators will match. Matching the terms in the numerator we find that we need,

$$\left(\frac{RD}{M} + \frac{L_e K}{M} + \frac{{K_f}^2}{M}\right) = \left(\frac{1}{C} + \frac{R}{Cr} + \frac{L_e}{LC}\right) \tag{33}$$

From which it follows that,

$$\frac{{K_f}^2}{M} = \frac{1}{C} \tag{34}$$

The normalised form of a second order polynomial in S is,
$S^2 + S\frac{\omega_o}{Q} + {\omega_o}^2$
We can produce a normalised form for the impedance if we introduce the terms ω_s as the resonant frequency[6] of the driver, Q_m as the mechanical quality factor for the driver, Q_e as the electrical quality factor and ω_e as the pole resulting from the leakage inductance L_e. Then we define,

$$\frac{\omega_s}{Q_m} = \frac{1}{Cr} = \frac{D}{M} \tag{35}$$

$$\frac{\omega_s}{Q_e} = \frac{K_f^2}{RM} \tag{36}$$

$${\omega_s}^2 = \frac{1}{LC} = \frac{K}{M} \tag{37}$$

$$\omega_e = \frac{R}{L_e} \tag{38}$$

Normalising our impedance function we find,

[6]Be aware that ω_s is the angular resonant frequency with units of radians per second. This relates to the Hertzian frequency through the relation $\omega_s = 2\pi f_s$ where f_s is the resonant frequency in Hz.

$$Z\left(S\right)=R\frac{S^{3}\left(\frac{1}{\omega_{e}}\right)+S^{2}\left(1+\frac{\omega_{s}}{Q_{m}\omega_{e}}\right)+S\left(\frac{\omega_{s}{}^{2}}{\omega_{e}}+\frac{\omega_{s}}{Q_{m}}+\frac{\omega_{s}}{Q_{e}}\right)+\omega_{s}{}^{2}}{S^{2}+S\left(\frac{\omega_{s}}{Q_{m}}\right)+\omega_{s}{}^{2}} \tag{39}$$

From equations 34 , 36 and 35 we can show that,

$$r=R\frac{Q_{m}}{Q_{e}} \tag{40}$$

Re-arranging equation 36 gives,

$$C=\frac{Q_{e}}{\omega_{s}R} \tag{41}$$

Finally, substituting C above into equation 37 and re-arranging gives,

$$L=\frac{R}{\omega_{s}Q_{e}} \tag{42}$$

If we Ignore the effect of L_e, which is typically small at driver resonance, we can simplify the imedance to,

$$Z\left(S\right)=R\left(\frac{S^{2}+S\left(\frac{\omega_{s}}{Q_{e}}+\frac{\omega_{s}}{Q_{m}}\right)+\omega_{s}{}^{2}}{S^{2}+S\left(\frac{\omega_{s}}{Q_{m}}\right)+\omega_{s}{}^{2}}\right) \tag{43}$$

From which we see that at the resonant frequency the magnitude of the driver impedance is at a maximum of,

$$\left|Z\left(j\omega_{s}\right)\right|=Z_{max}=R\frac{Q_{m}+Q_{e}}{Q_{e}} \tag{44}$$

This relationship comes in handy for determining the box Q of an enclosure, a parameter not readily predictable from theory.

Conjugate Load Matching

An important part of passive crossover design is the construction of a network to equalise the impedance of a driver so that it presents a constant resistive load to the crossover. Without equalisation the reactive driver impedance will result in a combined filter response that deviates from the desired response.

Consider the generalised equaliser of figure 5.

The input impedance of this network is,

$$Z_{in}=\frac{1}{\frac{1}{R+Z_{l}}+\frac{1}{R+Z_{c}}} \tag{45}$$

$$Z_{in}=\frac{R^{2}+Z_{l}Z_{c}+R\left(Z_{l}+Z_{c}\right)}{2R+Z_{l}+Z_{c}} \tag{46}$$

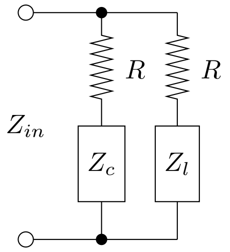


Figure 5: Generalised Conjugate Impedance

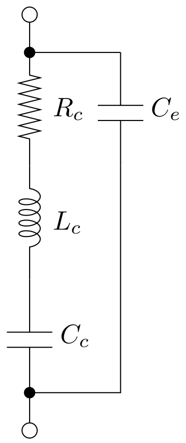


Figure 6: Impedance Equaliser Network Z_c

where Z_l is the complex impedance we wish to equalise and Z_c is the compensation impedance. We want Z_{in} to be R from which it follows that for this to be true then,

$$Z_c = \frac{R^2}{Z_l} = R^2 Y_l \tag{47}$$

where Y_l is the load admittance. To see how this translates into an equalisation for the loudspeaker impedance, consider the network in figure 6.

The impedance of this network is,

$$Z_c = \frac{1}{\frac{1}{SC_e} + \frac{1}{R_c} + SL_c + \frac{1}{SC_c}} \tag{48}$$

$$Z_c = \frac{R_c + SL_c + \frac{1}{SC_c}}{SC_e\left(R_c + SL_c + \frac{1}{SC_c}\right) + 1} \tag{49}$$

From equations 30 and 47 we can see that,

$$R^2 Y_l = \frac{\frac{R^2}{SL} + \frac{R^2}{r} + SCR^2}{SL_e\left(\frac{1}{SL} + \frac{1}{r} + SC\right) + 1} \tag{50}$$

Comparing the top lines of equation 49 with equation 50 we see that,

$$R_c = \frac{R^2}{r} \tag{51}$$

$$L_c = CR^2 \tag{52}$$

$$C_c = \frac{L}{R^2} \tag{53}$$

Comparing the bottom lines of equation 49 with equation 50 we see that,

$$\frac{C_e}{C_c} = \frac{L_e}{L} \tag{54}$$

$$C_e R_c = \frac{L_e}{r} \tag{55}$$

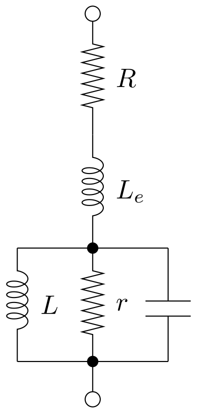
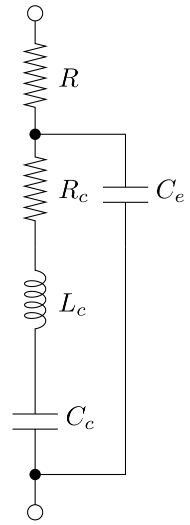
$$C_e L_c = L_e C \tag{56}$$

This all holds true if,

$$C_e = \frac{L_e}{R^2} \tag{57}$$

$$\tag{58}$$

Summarising the results we have,

Model impedance component values	Conjugate load match component values
	
$r = R\frac{Q_m}{Q_e}$	$R_c = \frac{R^2}{r} = R\frac{Q_e}{Q_m}$
$C = \frac{Q_e}{\omega_s R}$	$C_c = \frac{L}{R^2} = \frac{1}{\omega_s Q_e R}$
$L = \frac{R}{\omega_s Q_e}$	$L_c = CR^2 = \frac{Q_e R}{\omega_s}$
	$C_e = \frac{L_e}{R^2}$

The Response of a Driver in a Sealed Box

Mounting a driver in a sealed box adds two extra components to the mechanical model of figure 1: an extra damping term, D_b, and an extra stiffness term, K_b. These extra terms act in parallel with the corresponding driver terms so in essence, a sealed box simply increases the driver suspension stiffness and the driver mechanical damping. Referring to equation 17 and 19 we can see that,

$$\frac{D}{M} \Rightarrow \frac{D}{M} + \frac{D_b}{M} = \frac{\omega_s}{Q_m} + \frac{\omega_b}{Q_b} \tag{59}$$

and

$$\frac{K}{M} \Rightarrow \frac{K + Kb}{M} = {\omega_s}^2 \left(1 + \frac{V_{AS}}{V_b}\right) = {\omega_b}^2 \tag{60}$$

The normalised form of the K_b term stems from the definition of V_{AS}, the volume equivalent stiffness of the driver. By definition,

$$K_b = K\frac{V_{AS}}{V_b} \tag{61}$$

Equation 60 shows that box raises the resonant frequency of the driver with the new resonance becoming.

$$\omega_b = \omega_s\sqrt{1+\frac{V_{AS}}{V_b}} \tag{62}$$

or put in another way, the box volume required to raise the resonance to ω_b is,

$$V_b = \frac{V_{AS}}{\left(\frac{\omega_b}{\omega_s}\right)^2 - 1} \tag{63}$$

The new displacement and sound pressure responses are then,

$$\frac{X(S)}{V(S)} = \left(\frac{1}{K_f\omega_s Q_e}\right)\frac{{\omega_s}^2}{S^2 + \left(\frac{\omega_s}{Q_m}+\frac{\omega_s}{Q_e}+\frac{\omega_b}{Q_b}\right)S + {\omega_b}^2} \tag{64}$$

and

$$\frac{A_{spl}(S)}{V(S)} = \left(\frac{K_a\omega_s}{K_f Q_e}\right)\frac{S^2}{S^2 + \left(\frac{\omega_s}{Q_m}+\frac{\omega_s}{Q_e}+\frac{\omega_b}{Q_b}\right)S + {\omega_b}^2} \tag{65}$$

We note that the box has no effect on the sensitivity of the response above resonance but the displacement response below resonance is reduced in magnitude by the factor $\frac{{\omega_s}^2}{{\omega_b}^2}$, by virtue of the extra stiffness that the box provides.

The amount of extra damping that the enclosure provides is not readily predictable and will depend on the amount of fibrous lining inside the cabinet and no doubt, cabinet shape and volume. A reasonable estimate could be a box Q of around 7, though anyone wanting to obtain the optimum alignment should measure the impedance of the driver mounted in the box and determine Q_b via equation 44 and the manufacturer supplied Q_m and Q_b values.

For the sake of simplicity, if we assume that the box provides only added stiffness and no extra damping, then the net effect of the box is to raise the resonant frequency, but in raising the resonant frequency we change the Q_t because[7],

$$\frac{A_{spl}(S)}{V(S)} = \left(\frac{K_a\omega_s}{K_f Q_e}\right)\frac{S^2}{S^2 + \left(\frac{\omega_s}{\frac{\omega_b}{\omega_s}Q_t}\right)S + {\omega_b}^2} \tag{66}$$

Hence,

$$Q_{inbox} = \frac{\omega_b}{\omega_s}Q_t \tag{67}$$

Therefore to obtain an ideal Butterworth response with a given driver in a sealed box, we need to increase the inbox Q to a value of 0.707 (the Q for a second order

[7] Q_t is the total driver Q and is given by $\frac{1}{Q_t} = \frac{1}{Q_m} + \frac{1}{Q_e}$

Butterworth polynomial). As an example consider the driver whose parameters are,

F_s	57
Q_t	0.59
V_{AS}	8.77 litres

We see that the driver resonance needs to be increased by the factor $0.707/0.59 = 1.2$ times to 68 Hz. This implies we need a box volume of $8.77/(1.2^2 - 1) = 20$ litres. In practise, because the box provides additional damping, a box volume of 20 litres will be too large and the overall Q in the box too low. The additional box damping has the effect of making the driver Q_t lower, hence requiring a greater resonant frequency ratio.

The Response of a Driver in a Vented Box

7 shows a mechanical model of a loudspeaker in a vented enclosure. As with the driver model, M is the cone moving mass, K the suspension stiffness and D the driver mechanical damping. The box volume acts as a spring with stiffness K_b and is coupled to a moving mass M_v : the mass of air in the vent. A damping term associated with mechanical energy losses in the box is included (D_b) and is shown coupling to ground. A more complete model could be constructed by adding an additional damping term that couples M to M_v but it greatly increases the complexity of the algebra with not much improvement in model reality, so I have deliberately chosen not to include it.

Balancing the forces in this dynamic system we find,

$$M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + Kx + D\frac{dx}{dt} + (x - x_v)\,K_b = F \tag{68}$$

$$M_v\frac{d^2x_v}{dt^2} + D_b\frac{dx_v}{dt} + (x_v - x)\,K_b = 0 \tag{69}$$

Transforming to the complex frequency domain we find,

$$\frac{F(S)}{X(S)} = MS^2 + DS + (K + K_b) - \frac{X_v(S)}{X(S)}K_b \tag{70}$$

$$\frac{X(S)}{X_v(S)} = \frac{M_vS^2 + D_bS + K_b}{K_b} \tag{71}$$

Elimination $X_v(S)$ we find,

$\frac{F(S)}{X(S)} = MS^2 + DS + (K + K_b) - \frac{{K_b}^2}{M_vS^2 + D_bS + K_b}$

$$\frac{F(S)}{X(S)} = \frac{A_4S^4 + A_3S^3 + A_2S^2 + A_1S + A_0}{M_vS^2 + D_bS + K_b} \tag{72}$$

where,

$A_4 = MM_v$

$A_3 = M_vD_b + M_vD$

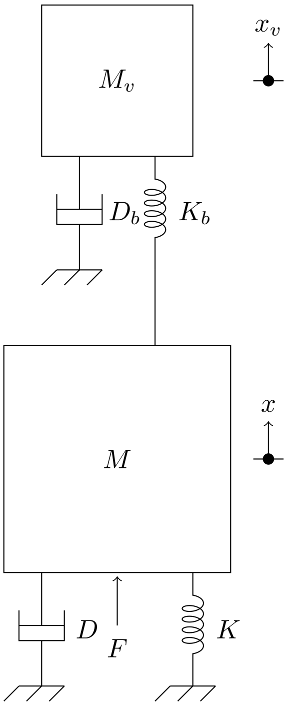


Figure 7: Mechanical Model of a Driver in a Vented Box

$A_2 = M_v K + M_v K_b + DD_b + MK_b$
$A_1 = K_b D + KD_b + K_b D_b$
$A_0 = KK_b$
Now substituting into equation 9 we find,

$$\frac{X(S)}{V(S)} = K_f \frac{M_v S^2 + D_b S + K_b}{B_5 S^5 + B_4 S^4 + B_3 S^3 + B_2 S^2 + B_1 S + B_0} \tag{73}$$

where,
$B_5 = MM_v L_e$
$B_4 = MM_v R + MD_b L_e + M_v D L_e$
$B_3 = MD_b R + M_v DR + M_v K L_e + M_v K_b L_e + DD_b L_e + MK_b L_e + M_v {K_f}^2$
$B_2 = M_v KR + M_v K_b R + DD_b R + MK_b R + K_b D L_e + KD_b L_e + K_b D_b L_e + D_b {K_f}^2$
$B_1 = K_b DR + KD_b R + K_b D_b R + K_b {K_f}^2$
$B_0 = KK_b R$
Or in a normalised form,

$$\frac{X(S)}{V(S)} = \left(\frac{1}{K_f \omega_s Q_e}\right) \frac{{\omega_s}^2 S^2 + \frac{\omega_b {\omega_s}^2}{Q_b} S + {\omega_b}^2 {\omega_s}^2}{N_5 S^5 + N_4 S^4 + N_3 S^3 + N_2 S^2 + N_1 S + N_0} \tag{74}$$

where,
$N_5 = \frac{1}{\omega_e}$

$N_4 = 1 + \frac{\omega_b}{Q_b\omega_e} + \frac{\omega_s}{Q_m\omega_e}$
$N_3 = \frac{\omega_b}{Q_b} + \frac{\omega_s}{Q_m} + \frac{{\omega_s}^2}{\omega_e}\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_s\omega_b}{Q_mQ_b\omega_e} + \frac{{\omega_b}^2}{\omega_e} + \frac{\omega_s}{Q_e}$
$N_2 = {\omega_s}^2\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_s\omega_b}{Q_mQ_b} + {\omega_b}^2 + \frac{{\omega_b}^2\omega_s}{Q_m\omega_e} + \frac{{\omega_s}^2\omega_b}{\omega_eQ_b}\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_b\omega_s}{Q_bQ_e}$
$N_1 = \frac{{\omega_b}^2\omega_s}{Q_m} + \frac{{\omega_s}^2\omega_b}{Q_b}\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{{\omega_b}^2\omega_s}{Q_e}$
$N_0 = {\omega_s}^2{\omega_b}^2$
$\omega_e = \frac{R}{L_e}$
$\frac{\omega_b}{Q_b} = \frac{D_b}{M_v}$
$\frac{\omega_s}{Q_s} = \frac{D}{M}$
${\omega_b}^2 = \frac{K_b}{M_v}$
${\omega_s}^2 = \frac{K}{M}$
If we ignore the effect of L_e then the simplified displace becomes,

$$\frac{X(S)}{V(S)} = \left(\frac{1}{K_f\omega_sQ_e}\right)\frac{{\omega_s}^2S^2 + \frac{\omega_b{\omega_s}^2}{Q_b}S + {\omega_b}^2{\omega_s}^2}{S^4 + N_3S^3 + N_2S^2 + N_1S + N_0} \tag{75}$$

where,
$N_3 = \frac{\omega_b}{Q_b} + \frac{\omega_s}{Q_m}$
$N_2 = {\omega_s}^2\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_s\omega_b}{Q_mQ_b} + {\omega_b}^2 + \frac{\omega_b\omega_s}{Q_bQ_e}$
$N_1 = \frac{{\omega_b}^2\omega_s}{Q_m} + \frac{{\omega_s}^2\omega_b}{Q_b}\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{{\omega_b}^2\omega_s}{Q_e}$
$N_0 = {\omega_s}^2{\omega_b}^2$
The vent displacement response follows from,

$$\frac{X_v(S)}{V(S)} = \frac{X(S)}{V(S)}\frac{X_v(S)}{X(S)} \tag{76}$$

Thus,

$$\frac{X_v(S)}{V(S)} = \left(\frac{1}{K_f\omega_sQ_e}\right)\frac{{\omega_b}^2{\omega_s}^2}{S^4 + N_3S^3 + N_2S^2 + N_1S + N_0} \tag{77}$$

The sound pressure response components of the driver and the vent are found by multiplying each by K_aS^2. The radiated sound pressure is the difference between the two (remember that the vent is driven by the back side of the driver). Therefore,

$$\frac{A_{SPL}(S)}{V(S)} = \left(\frac{K_a\omega_s}{K_fQ_e}\right)\frac{S^4 + \frac{\omega_b}{Q_b}S^3}{S^4 + N_3S^3 + N_2S^2 + N_1S + N_0} \tag{78}$$

We see that the sound pressure response is a fourth order high pass response as opposed to a second order high pass response for a sealed box enclosure. The cutoff slope is therefore asymptotic to 24 dB per octave.

To determine the driver impedance mounted in a vented box we substitute the displacement response of equation 73 into the generalised impedance of equation 26 giving,

$$Z(S) = (R + SL_e)\frac{B_5S^5 + B_4S^4 + B_3S^3 + B_2S^2 + B_1S + B_0}{B_5S^5 + B_4S^4 + B_8S^3 + B_7S^2 + B_6S + B_0} \tag{79}$$

where,

$B_8 = MD_bR + M_vDR + M_vKL_e + M_vK_bL_e + DD_bL_e + MK_bL_e$

$B_7 = M_vKR + M_vK_bR + DD_bR + MK_bR + K_bDL_e + KD_bL_e + K_bD_bL_e$

$B_6 = K_bDR + KD_bR + K_bD_bR$

or in normalised form,

$$Z(S) = (R + SL_e)\frac{N_5S^5 + N_4S^4 + N_3S^3 + N_2S^2 + N_1S + N_0}{N_5S^5 + N_4S^4 + N_8S^3 + N_7S^2 + N_6S + N_0} \tag{80}$$

where,

$N_8 = \frac{\omega_b}{Q_b} + \frac{\omega_s}{Q_m} + \frac{{\omega_s}^2}{\omega_e}\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_s\omega_b}{Q_mQ_b\omega_e} + \frac{{\omega_b}^2}{\omega_e}$

$N_7 = {\omega_s}^2\left(1 + \frac{V_{AS}}{V_b}\right) + \frac{\omega_s\omega_b}{Q_mQ_b} + {\omega_b}^2 + \frac{{\omega_b}^2\omega_s}{Q_m\omega_e} + \frac{{\omega_s}^2\omega_b}{Q_b\omega_e}\left(1 + \frac{V_{AS}}{V_b}\right)$

$N_6 = \frac{{\omega_b}^2\omega_s}{Q_m} + \frac{{\omega_s}^2\omega_b}{Q_b}\left(1 + \frac{V_{AS}}{V_b}\right)$

By measuring the impedance of the driver mounted in the prototype cabinet, we can verify the box tuning and if not optimal, adjust it accordingly, usually by altering the box frequency (by changing the length of the vent) and changing the box Q (by adding or removing damping material in the form of box stuffing). In this way we can obtain the best possible performance from our system even though we cannot accurately measure the frequency response itself (standing waves colour any attempt for the home builder to accurately measure the real bass response of the enclosure directly).

$$\tag{81}$$

The Vent Length

A vent in a sealed box acts as a resonator whose resonance is controlled by the stiffness of the air spring and and mass of the moving air within the vent. This general type of acoustic system is referred to as a Helmholtz resonator in the literature. The theoretical stiffness of the air spring is governed by the box volume and the cross sectional area of the vent. Without going into the detail of the derivation, it can be shown that,

$$l_{vent} \approx 23318\left(\frac{d^2}{V_b{f_b}^2}\right) \tag{82}$$

where the length and diameter is measure in centimetres, the box frequency in Hertz and the box volume in litres.

In practice the real length required to obtain the desired tuning is shorter than the above due to end effects. For real vents there is an extra portion of moving mass extending out from either end of the vent. The length extension is approximately $0.85d$ for flanged vent ends and $0.6d$ for unflanged ends. In a typical box construction we have one flanged and one unflanged end so the length extension is in the order of $0.725d$. Hence the actual cut length for the vent is more typically,

$$l_{vent} \approx 23318 \left(\frac{d^2}{V_b {f_b}^2} \right) - 0.725d \tag{83}$$

though we should verify the box tuning by measuring the driver impedance. The driver mounted in a vented box will display two resonances, the upper anti-phase mode and the lower in-phase mode. The box frequency is approximately the frequency at which the impedance is at a minimum between these two peaks.